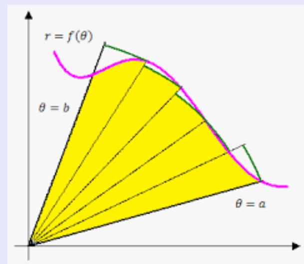
Calculus II

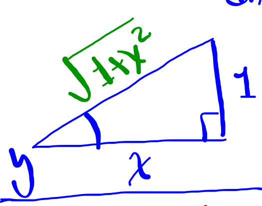
Lecture 5



Feb 19-8:47 AM

Class QZ 5 (Open Notes)

Show $\frac{d}{dx}\left[\cot^{-1} x\right] = \frac{-1}{1+x^2}$



$y = \cot^{-1} x$

$x = \cot y$

$1 = -\csc^2 y \frac{dy}{dx}$

$$\boxed{\sin y = \frac{1}{\sqrt{1+x^2}}}$$

$\frac{dy}{dx} = \frac{-1}{\csc^2 y}$

$\sin^2 y = \frac{1}{1+x^2}$

$$\boxed{\frac{d}{dx}\left[\cot^{-1} x\right] = \frac{-1}{1+x^2}}$$

$\frac{dy}{dx} = -\sin^2 y$

Find $\sinh 0$.

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\sinh 0 = \frac{e^{0} - e^{0}}{2} = \frac{1-1}{2} = \boxed{0}$$

Find $\cosh(\ln 3)$

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$\cosh(\ln 3) = \frac{e^{\ln 3} + e^{-\ln 3}}{2} = \frac{3 + 3^{-1}}{2}$$

$$= \frac{3 + \frac{1}{3}}{2} = \frac{9+1}{6} = \frac{10}{6} = \boxed{\frac{5}{3}}$$

Find $\sinh^{-1} 1$

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\sinh^{-1} 1 = \ln\left(1 + \sqrt{1^2+1}\right) = \boxed{\ln\left(1+\sqrt{2}\right)}$$

Prove $\tanh(x+y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

$$\tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

$$\tanh y = \frac{e^{y} - e^{-y}}{e^{y} + e^{-y}}$$

$$\Rightarrow \frac{\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} + \frac{e^{y}-e^{-y}}{e^{y}+e^{-y}}}{1 + \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} \cdot \frac{e^{y}-e^{-y}}{e^{y}+e^{-y}}}$$

$$\text{LCD} = (e^{x}+e^{-x})(e^{y}+e^{-y})$$

$$= \frac{(e^{y}+e^{-y})(e^{x}-e^{-x}) + (e^{x}+e^{-x})(e^{y}-e^{-y})}{(e^{x}+e^{-x})(e^{y}+e^{-y}) + (e^{x}-e^{-x})(e^{y}-e^{-y})}$$

$$= \frac{e^{x+y} \cancel{- e^{y-x}} \cancel{+ e^{-y+x}} - e^{-y-x} + e^{x+y} \cancel{- e^{x-y}} \cancel{+ e^{-x+y}} - e^{-x-y}}{e^{x+y} \cancel{+ e^{x-y}} \cancel{+ e^{y-x}} + e^{-x-y} + e^{x+y} \cancel{- e^{x-y}} \cancel{- e^{-x+y}} + e^{-x-y}}$$

$$= \frac{2e^{x+y} - 2e^{-(x+y)}}{2e^{x+y} + 2e^{-(x+y)}} = \frac{e^{x+y} - e^{-(x+y)}}{e^{x+y} + e^{-(x+y)}}$$

$$= \tanh(x+y)$$

$$\frac{d}{dx}\left[\tanh\left(1+e^{2x}\right)\right]$$

$$\frac{d}{dx}\left[\tanh u\right] = \operatorname{sech}^2 u \cdot \frac{du}{dx}$$

$$= \operatorname{sech}^2\left(1+e^{2x}\right)\cdot\left[0+2e^{2x}\right]$$

$$= 2e^{2x}\operatorname{sech}^2\left(1+e^{2x}\right)$$

$$\frac{d}{dx}\left[\tanh^{-1} e^{x}\right]$$

$$\frac{d}{dx}\left[\tanh^{-1} u\right] = \frac{1}{1-u^2}\cdot\frac{du}{dx}$$

$$= \frac{1}{1-(e^x)^2}\cdot e^x$$

$$= \frac{e^x}{1-e^{2x}}$$

$$\int \sinh(1+4x)\,dx$$

Let $u = 1+4x$

$du = 4\,dx$

$\frac{du}{4} = dx$

$$= \int \sinh u \,\frac{du}{4}$$

$$= \frac{1}{4}\int \sinh u\,du$$

Recall

$$\frac{d}{dx}\left[\cosh x\right] = \sinh x$$

$$= \frac{1}{4}\cosh u + C = \boxed{\frac{1}{4}\cosh(1+4x) + C}$$

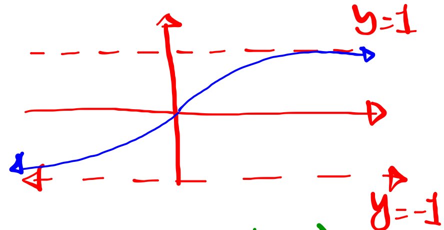
$$\int \frac{\operatorname{sech}^2 x}{2+\tanh x}\,dx \qquad u = 2+\tanh x$$

$$du = \operatorname{sech}^2 x\,dx$$

$$= \int \frac{1}{u}\,du = \ln|u| + C$$

$$= \ln|2+\tanh x| + C$$

Graph $y = \tanh x$



$$-1 < \tanh x < 1$$

$$1 < 2+\tanh x < 3$$

$$= \ln(2+\tanh x) + C$$

Find $\displaystyle\int \frac{e^x}{1-e^{2x}}\,dx$

$$= \int \frac{e^x}{1-(e^x)^2}\,dx \qquad \text{Let } u = e^x$$

$$du = e^x\,dx$$

$$= \int \frac{1}{1-u^2}\,du \qquad \frac{d}{dx}\left[\tanh^{-1} x\right] = \frac{1}{1-x^2}$$

$$= \tanh^{-1} u + C = \tanh^{-1} e^x + C$$